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Reduced determinantal forms for characters of the classical Lie groups

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Abstract. The reduced determinantal forms for the character of all the irreducible representations of the classical Lie groups are derived from Weyl's character formulae. The derivations use a lemma due to Frobenius and the results are expressed in terms of characters of representations specified by hook diagrams. The results are appropriate to all irreducible representations whether specified by ordinary or composite Young diagrams, and whether tensor or spinor in nature.

1. Introduction

In a definitive study of semi-simple Lie groups Weyl (1925, 1926) gave determinantal expansions of the characters of all the irreducible representations of the classical groups SU(N), SO(N) and Sp(N). These expansions expressed the characters in terms of complete homogeneous symmetric functions of the eigenvalues, $\exp(\mathrm{i}\phi_n)$, of the group element under consideration. The various labels ϕ_n, $n = 1, 2, \ldots$ serve to specify the corresponding conjugacy classes.

In the case of the unitary groups, SU(N), Littlewood (1940) gave an alternative determinantal expansion of the characters expressed in terms of elementary symmetric functions and went on to derive a reduced determinantal expansion involving hook functions. This was re-derived in a much more elegant manner by Foulkes (1951).

More recently (Abramsky, Jahn and King 1973), analogous reduced determinantal expansions were rather painfully derived for the characters of the tensor irreducible representations of the orthogonal groups O(N) and the symplectic groups Sp(N). The simplicity of the final results belied the nature of the derivation and it is the purpose of this paper to show that the previously overlooked paper by Foulkes (1951) affords a natural generalisation from U(N) to both O(N) and Sp(N).

The key to the derivation lies in an important lemma due to Frobenius (1900, 1903) which was also obtained and used very effectively by Foulkes (1951). This is proved very easily through a consideration of the Young diagrams specified by the partitions which label characters of irreducible representations. The notation appropriate to partitions and Young diagrams is given in § 2 along with a proof of Frobenius' lemma. In § 3 the derivation of the reduced determinantal expansions is carried out for all the invariant tensor representations of U(N), O(N) and Sp(N) starting from the character formulae of Weyl. The mixed tensor representations of U(N) specified by composite

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Young diagrams are dealt with in § 4 and in the concluding section the spinor characters of O(N) are discussed and an account is given of the characters of SO(N) which involves expansions not only of ordinary characters but also of difference characters.

2. Frobenius' lemma

A partition is denoted by $(\boldsymbol{\lambda}) = (\lambda_1, \lambda_2, \ldots)$ with λ_i a non-negative integer for $i = 1, 2, \ldots$ and $\lambda_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots$. This partition specifies a Young diagram consisting of left-adjusted rows of boxes such that the ith row contains λ_i boxes. This is illustrated in figure 1 for the case $(\boldsymbol{\lambda}) = (4\ 3\ 2^2\ 1)$.

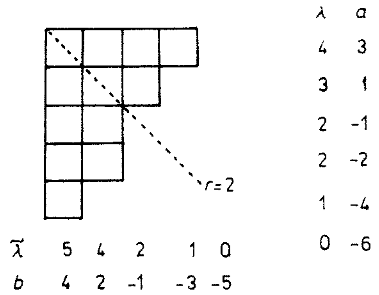

Figure 1. Young diagrams and the Frobenius notation. $a_i = \lambda_i - i$ and $b_j = \tilde{\lambda}_j - j$ for $i, j = 1, 2, \ldots$; $a_i \geq 0$ and $b_j \geq 0$ for $i, j = 1, 2, \ldots, r$; $a_i < 0$ and $b_j < 0$ for $i, j = r+1, r+2 \ldots$;

$$(\boldsymbol{\lambda}) = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r \end{pmatrix}.$$

The partition $(\tilde{\boldsymbol{\lambda}}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ conjugate to $(\boldsymbol{\lambda})$ is defined in such a way that $\tilde{\lambda}_j$ is the number of boxes in the jth column of the Young diagram specified by $(\boldsymbol{\lambda})$. Thus if $(\boldsymbol{\lambda}) = (4\ 3\ 2^2\ 1)$ then $(\tilde{\boldsymbol{\lambda}}) = (5\ 4\ 2\ 1)$.

It is sometimes convenient to adopt the notation for a partition due to Frobenius (1900). This notation is such that if $a_i = \lambda_i - i$ for $i = 1, 2, \ldots$ and $b_j = \tilde{\lambda}_j - j$ for $j = 1, 2, \ldots$. then the partition $(\boldsymbol{\lambda})$ is denoted by

$$(\boldsymbol{\lambda}) = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r \end{pmatrix} \tag{2.1}$$

where r, the Frobenius rank of the partition, is the number of boxes on the main diagonal of the Young diagram specified by $\boldsymbol{\lambda}$. Clearly

$$a_i > a_{i+1} \text{ for } i = 1, 2, \ldots, \qquad b_j > b_{j+1} \quad \text{for} \quad j = 1, 2, \ldots \tag{2.2}$$

and

$$\begin{aligned} &a_i \geq 0 \text{ for } i \leq r, \qquad && b_j \geq 0 \text{ for } j \leq r, \\ &a_i < 0 \text{ for } i > r, \qquad && b_j < 0 \text{ for } j > r. \end{aligned} \tag{2.3}$$

The notation is exemplified in the case of the partition $(\boldsymbol{\lambda}) = (4\ 3\ 2^2\ 1)$ and its conjugate

$(\boldsymbol{\lambda}) = (5\ 4\ 2\ 1)$ by

$$(4\ 3\ 2^2\ 1) = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \quad \text{and} \quad (5\ 4\ 2\ 1) = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$$

with $r = 2$ in both cases. In the case of $(\boldsymbol{\lambda}) = (4\ 3\ 2^2\ 1)$ the sequences a_i and b_i are $(3, 1, -1, -2, -4, -6, -7, -8, \ldots)$ and $(4, 2, -1, -3, -5, -6, -7, -8, \ldots)$ as indicated in figure 1.

The following lemma due to Frobenius (1900 § 6 (4), 1903 § 5 (5)) is of great use in studying the characters of the representations of the classical groups specified by partitions. It was pointed out by Foulkes (1951) that its proof follows directly from a consideration of the Young diagrams specified by partitions.

Frobenius' lemma. The sets $\{a_i\colon i = 1, 2, \ldots\}$ and $\{-1 - b_j\colon j = 1, 2, \ldots\}$ are non-intersecting.

Proof. Consider the point in the ith row and jth column of the Young diagram specified by $(\boldsymbol{\lambda}) = \binom{a}{b}$. If this point lies inside the Young diagram, i.e. there is a box in the ijth position, then the hook length (Robinson 1961 p 44) defined by

$$\begin{aligned} h_{ij} &= (\lambda_i - j) + (\tilde{\lambda}_j - i) + 1 \\ &= a_i + b_j + 1 \end{aligned} \tag{2.4}$$

is positive. If the point lies outside the Young diagram, i.e. there is no box in the ijth position, then the hook length is negative. This is made clear in figure 2. Hence for all i and j either $h_{ij} > 0$ or $h_{ij} < 0$, i.e. $h_{ij} \neq 0$, so that $a_i \neq -1 - b_j$ for any i and j. This completes the proof.

The result is illustrated in the case $(\boldsymbol{\lambda}) = (4\ 3\ 2^2\ 1)$ by the sets $\{3, 1, -1, -2, -4, -6, -7, -8, \ldots\}$ and $\{-5, -3, 0, 2, 4, 5, 6, \ldots\}$.

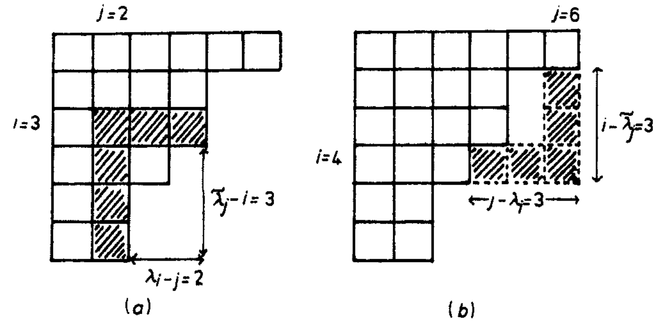

Figure 2. The properties of hook lengths. $\boldsymbol{\lambda} = (6\ 4^2\ 3\ 2^2)$. $h_{ij} = \lambda_i - j + \tilde{\lambda}_j - i + 1 = a_i + b_j + 1$ (i) If the ith row and jth column intersect then $h_{ij} > 0$, (ii) If the ith row and jth column do not intersect then $h_{ij} < 0$. (a) $h_{32} = 2 + 3 + 1 = 6 > 0$. ($b$) $h_{46} = -3 - 3 + 1 = -5 < 0$.

3. Hook character expansions

The characters of the irreducible representations of the unitary group $U(N)$ may be conveniently denoted by $\{\boldsymbol{\lambda}\}$ where $(\boldsymbol{\lambda})$ is a partition. It is well known that these characters possess a determinantal expansion given by Weyl (1925 § 6 (40)):

$$\{\boldsymbol{\lambda}\} = |\{\lambda_i - i + j\}| \tag{3.1}$$

where the ijth element of the determinant has been displayed and is the character associated with a Young diagram consisting of a single row. It is in fact a complete homogeneous symmetric function of the characteristic roots of the group element under consideration. Similarly Littlewood (1940 p 89) has given the expansion

$$\{\boldsymbol{\lambda}\} = |\{1^{\tilde{\lambda}_i+i-j}\}| \tag{3.2}$$

where the ijth element of the determinant is the character associated with a Young diagram consisting of a single column. This character is an elementary symmetric function of the characteristic roots of the group element.

In interpreting (3.1) and (3.2) it is important to note that the indices i and j may be taken to range over the values $1, 2, \ldots, n$ for any n such that $n \geqslant \tilde{\lambda}_1$ and $n \geqslant \lambda_1$ respectively. This is easily seen to follow from the fact that

$$\{m\} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases} \tag{3.3}$$

and

$$\{1^m\} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m < 0. \end{cases} \tag{3.4}$$

Foulkes (1951) pointed out essentially, that,

$$|(-1)^{i+1}\{1^{n+1-i-j}\}| = 1$$

by virtue of (3.4). Together with (3.1) this result yields, by taking the determinant of a product of matrices, the expansion

$$\{\boldsymbol{\lambda}\} = \left| \sum_{k=1}^{n} (-1)^{k+1}\{\lambda_i - i + k\}\{1^{n+1-k-j}\} \right|. \tag{3.5}$$

The application of (3.1) to the character $\{\boldsymbol{\mu}\}$, with $\mu_1 = \lambda_i - i + 1$, $\mu_2 = \mu_3 = \ldots = \mu_{n-j+1} = 1$, $\mu_{n-j+2} = \mu_{n-j+3} = \ldots = \mu_n = 0$, gives, after expanding with respect to the elements of the first row and rewriting the minors as characters of the form $\{1^m\}$ using (3.1) again, the result

$$\{\lambda_i - i + 1, 1^{n-j}\} = \sum_{k=1}^{n} (-1)^{k+1}\{\lambda_i - i + k\}\{1^{n+1-k-j}\}. \tag{3.6}$$

This result is valid even if $\lambda_i - i$ is negative in that

$$\{\lambda_i - i + 1, 1^{n-j}\} = \begin{cases} (-1)^{n-j} & \text{if } \lambda_i - i = -(n-j+1), \\ 0 & \text{if } \lambda_i - i < 0 \text{ and } \lambda_i - i \neq -(n-j+1), \end{cases} \tag{3.7}$$

as may be seen by examining the determinantal form (3.1) of the particular character $\{\boldsymbol{\mu}\}$.

Hence from (3.5) and (3.6)

$$\{\boldsymbol{\lambda}\} = |\{\lambda_i - i + 1, 1^{n-j}\}|. \tag{3.8}$$

The structure of this determinant has been discussed by Foulkes (1951) who extolled the merits of using the Frobenius notation.

Thus in (3.8) the elements of the first r rows consist of characters of the form $\{a_i + 1, 1^{n-j}\}$ with $a_i \geqslant 0$, whilst in the remaining rows for which $i > r$ there exists only one non-vanishing element and this appears, by (3.7), in the jth column with $j = n+1+a_i$. This entry is simply $(-1)^{n-i}$. Moreover the column labels of the determinant are furnished by the set

$$S = \{j : j = 1, 2, \ldots, n\},$$

whilst those columns containing the entries $(-1)^{n-i}$ are labelled by the subset

$$S_a = \{(n+1+a_i) : i = r+1, r+2, \ldots, n\}.$$

However the set

$$S_b = \{(n-b_k) : k = 1, 2, \ldots, r\}$$

is also a subset of S and by Frobenius' lemma the intersection of S_a and S_b is empty. Since the total number of elements in S_a and S_b is n, which is the number of elements in S, it follows that S is the disjoint union of S_a and S_b. The leads to the conclusion that an expansion of the determinant (3.8) with respect to the rows $r+1, r+2, \ldots, n$ leaves a single $r \times r$ determinant whose entries are of the form $\{a_i+1, 1^{n-j}\}$ with $i = 1, 2, \ldots, r$ and j a member of the set S_b. The overall sign factor arising from the terms $(-1)^{n-i}$ and the cofactor sign $(-1)^{r+1+j}$ is simply

$$\prod_{i=r+1}^{n} (-1)^{n+r+1} = (-1)^{(n+r+1)(n-r)} = +1 \qquad \forall n \text{ and } \forall r. \tag{3.9}$$

Thus in terms of the Frobenius partition parameters

$$\{\boldsymbol{\lambda}\} = \begin{Bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{Bmatrix} = \left| \begin{Bmatrix} a_i \\ b_j \end{Bmatrix} \right| \tag{3.10}$$

in which the ijth element of the determinant is the character $\{a_i+1, 1^{b_j}\}$ of a representation of U(N) specified by a Young diagram consisting of a single hook of arm length a_i and leg length b_j, with i and j now ranging only over the values $1, 2, \ldots, r$.

The result was first derived by Littlewood (1940 p 12). When expressed in terms of the partition label $\boldsymbol{\lambda}$ it takes the form

$$\{\boldsymbol{\lambda}\} = |\{\lambda_i - i + 1, 1^{\tilde{\lambda}_j - j}\}|. \tag{3.11}$$

This remarkable result has been generalised to the case of the orthogonal groups O(N) and symplectic groups Sp(N) (Abramsky *et al* 1973), using an inductive argument. However the generalisation is achieved much more simply by using the determinantal expansions of the characters of these groups given by Weyl (1926 § 5 (37), (39) and § 3 (20)). The tensor characters of the full orthogonal group O(N) and the symplectic group Sp(N) are here denoted by $[\boldsymbol{\lambda}]$ and $\langle\boldsymbol{\lambda}\rangle$ respectively. With this notation

$$[\boldsymbol{\lambda}] = |\{\lambda_i - i + j\} - \{\lambda_i - i - j\}| \tag{3.12}$$

and

$$\langle\boldsymbol{\lambda}\rangle = |\{\lambda_i - i + j\} + (1 - \delta_{1j})\{\lambda_i - i - j + 2\}|. \tag{3.13}$$

Multiplication exactly as for the unitary group characters by the matrix corresponding to (3.5) then yields

$$[\boldsymbol{\lambda}] = \left| \sum_{k=1}^{n} (-1)^{k+1} (\{\lambda_i - i + k\} - \{\lambda_i - i - k\})\{1^{n+1-k-j}\} \right| \tag{3.14}$$

and

$$\langle\boldsymbol{\lambda}\rangle=\left|\sum_{k=1}^{n}(-1)^{k+1}(\{\lambda_i-i+k\}+(1-\delta_{1k})\{\lambda_i-i-k+2\})\{1^{n+1-k-j}\}\right|. \tag{3.15}$$

However, precisely as before the ijth elements of these determinants are nothing other than $[\lambda_i-i+1, 1^{n-j}]$ and $\langle\lambda_i-i+1, 1^{n-j}\rangle$ respectively. This may be verified directly from the application of the expansions (3.12) and (3.13) to these characters, expanding with respect to the first row of the resulting determinant and identifying the relevant minors as before. Additionally it follows from the application of (3.3) to (3.1), (3.12) and (3.13) that

$$\{\lambda_i-i+1, 1^{n-j}\}=[\lambda_i-i+1, 1^{n-j}]=\langle\lambda_i-i+1, 1^{n-j}\rangle \qquad \text{if } \lambda_i-i<0. \tag{3.16}$$

Thus the use of (3.7) and the argument preceding (3.10) leads immediately to the generalisations of (3.10), namely

$$[\boldsymbol{\lambda}]=\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}=\left|\begin{bmatrix}a_i\\ b_j\end{bmatrix}\right| \tag{3.17}$$

and

$$\langle\boldsymbol{\lambda}\rangle=\left\langle\begin{matrix}\boldsymbol{a}\\ \boldsymbol{b}\end{matrix}\right\rangle=\left|\left\langle\begin{matrix}a_i\\ b_j\end{matrix}\right\rangle\right|, \tag{3.18}$$

or equivalently

$$[\boldsymbol{\lambda}]=|[\lambda_i-i+1, 1^{\tilde{\lambda}_j-j}]| \tag{3.19}$$

and

$$\langle\boldsymbol{\lambda}\rangle=|\langle\lambda_i-i+1, 1^{\tilde{\lambda}_j-j}\rangle|. \tag{3.20}$$

4. Composite Young diagrams

Composite Young diagrams were introduced (King 1970, Abramsky and King 1970) to simplify procedures for dealing with irreducible representations of $\mathrm{U}(N)$ corresponding to mixed (covariant and contravariant) tensors. The character of such a representation is denoted by $\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\}$ where $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are each partitions which together specify a composite Young diagram. The diagram is obtained by adjoining back-to-back the Young diagrams specified by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ with the boxes in the rows of $\boldsymbol{\mu}$ left-adjusted as usual but with those in the rows of $\boldsymbol{\nu}$ right-adjusted to the same line and distinguished by the insertion of a dot in each such box. This is illustrated in figure 3 in the case of the character $\{\overline{4\,3\,1}; 2^2\,1\}$.

These characters are related to those discussed above through the definition

$$\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\}=\bar{\epsilon}^r\{\boldsymbol{\lambda}\} \tag{4.1}$$

where ϵ is the character of the representation of $\mathrm{U}(N)$ in which each element is mapped to its determinant and $\bar{\epsilon}$ is the contragredient character, so that it is just the inverse of this determinant, whilst r is any integer satisfying the condition $r \geq \nu_1$, and the partition $\boldsymbol{\lambda}$ is defined by

$$\lambda_i=\begin{cases}\mu_i+r & \text{for } i=1,2,\ldots,\tilde{\mu}_1\\ r & \text{for } i=\tilde{\mu}_1+1, \tilde{\mu}_2+2,\ldots,N-\tilde{\nu}_1\\ r-\nu_{N-i+1} & \text{for } i=N-\tilde{\nu}_1+1, N-\tilde{\nu}_2+2,\ldots,N.\end{cases} \tag{4.2}$$

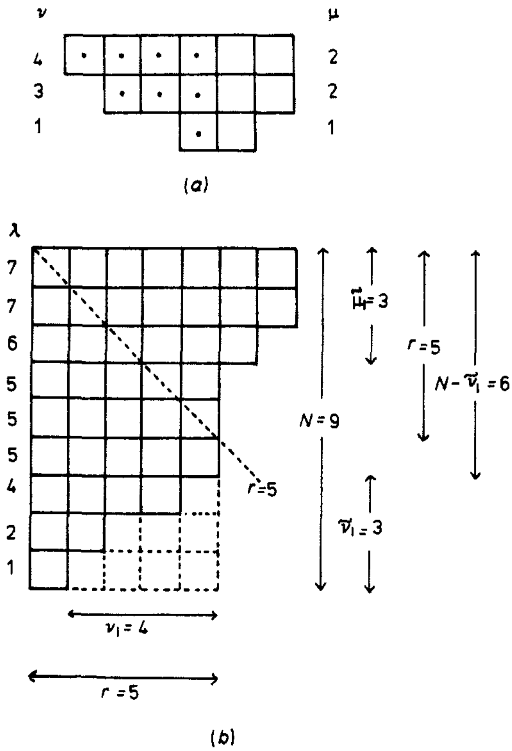

Figure 3. Composite Young diagrams and their connection with ordinary Young diagrams. $\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\} = \bar{\epsilon}^r\{\boldsymbol{\lambda}\}$ for the group U(N) with $N \geq \tilde{\mu}_1 + \tilde{\nu}_1$, $r \geq \nu_1$, $\tilde{\mu}_1 \leq r \leq N - \tilde{\nu}_1$. ($a$) $\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\} = \{\overline{4\,3\,1}; 2^2\,1\}$, ($b$) $\{\boldsymbol{\lambda}\} = \{7^2\,6\,5^3\,4\,2\,1\}$.

In the case of the group U(9) and the choice $r = 5$ this definition yields

$$\{\overline{4\,3\,1};\, 2^2\,1\} = \bar{\epsilon}^5\{7^2\,6\,5^3\,4\,2\,1\}$$

as illustrated in figure 3. It is clear that $(\boldsymbol{\lambda})$ specifies a Young diagram obtained from the composite Young diagram specified by $(\bar{\boldsymbol{\nu}}; \boldsymbol{\mu})$ by adjoining the complement of $(\boldsymbol{\nu})$ in (N^r) to $(\boldsymbol{\mu})$. This is such that the definition (4.2) could be replaced by the equivalent but simpler statement

$$\tilde{\lambda}_j = \begin{cases} N - \nu_{r-j+1} & \text{for } j = 1, 2, \ldots, r \\ \tilde{\mu}_{j-r} & \text{for } j = r+1, r+2, \ldots, r+\mu_1. \end{cases} \tag{4.3}$$

It is clear that $\boldsymbol{\lambda}$ is indeed a partition iff $N \geq \tilde{\mu}_1 + \tilde{\nu}_1$.

It is convenient in analysing the properties of the character $\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\}$ to use (4.1) in the case for which $r \geq \tilde{\mu}_1$ and $r \leq N - \tilde{\nu}_1$. For this to be possible it must be assumed that not only is $N \geq \tilde{\mu}_1 + \tilde{\nu}_1$ but also $N \geq \tilde{\mu}_1 + \nu_1$, since in (4.1) it is necessarily true that $r \geq \nu_1$. With this assumption r is simply the Frobenius rank of $\boldsymbol{\lambda}$. It then follows from (4.1), (4.2), (4.3) and (3.11) that

$$\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\} = \bar{\epsilon}^r |\{\mu_i - i + r + 1,\, 1^{N-\tilde{\nu}_{r-j+1}-j}\}| \tag{4.4}$$

with $i, j = 1, 2, \ldots, r$. However, a special case of (4.1) is simply

$$\{\overline{1^{\tilde{\nu}_1}};\, \mu_1\} = \bar{\epsilon}\{\mu_1 + 1,\, 1^{N-\tilde{\nu}_1-1}\} \tag{4.5}$$

so that

$$\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=|\overline{\{1^{\tilde{\nu}_{r-j+1}+j-1}};\mu_i-i+r\}|. \tag{4.6}$$

Reversing the order of the columns of this determinant and relabelling them using the transformation $j\to r-j+1$ then gives

$$\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=(-1)^{r(r-1)/2}|\overline{\{1^{\tilde{\nu}_j-j+r}};\mu_i-i+r\}| \tag{4.7}$$

where the ijth element of the determinant has been displayed and is the character specified by a composite Young diagram consisting of a single hook formed from a column of dotted boxes and a row of undotted boxes.

This result has been derived assuming that $N\geqslant\tilde{\mu}_1+\tilde{\nu}_1$ and $N\geqslant\tilde{\mu}_1+\nu_1$, but this restriction may now be dropped since the product rules for characters specified by composite Young diagrams, used in the evaluation of the determinant appearing in (4.7), are independent of N (King 1971). The remaining restrictions on r, independent of N, are $r\geqslant\tilde{\mu}_1$ and $r\geqslant\nu_1$, and since r may be arbitrarily increased without changing the validity of (4.7) it is convenient to replace it by the label m where

$$m=\max(\mu_1,\tilde{\mu}_1,\nu_1,\tilde{\nu}_1).$$

This avoids any possible confusion with the Frobenius rank of the partitions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. If the Frobenius notation

$$(\boldsymbol{\mu})=\begin{pmatrix}\boldsymbol{a}\\\boldsymbol{b}\end{pmatrix}\qquad\text{and }(\boldsymbol{\nu})=\begin{pmatrix}\boldsymbol{c}\\\boldsymbol{d}\end{pmatrix}$$

is adopted and extended so that

$$\begin{aligned}a_i&=\mu_i-i &&\text{and}\qquad b_j=\tilde{\mu}_j-j,\\ c_i&=\nu_i-i &&\text{and}\qquad d_j=\tilde{\nu}_j-j\end{aligned} \tag{4.8}$$

for $i,j=1,2,\ldots,m$, then (4.17) finally yields

$$\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=(-1)^{m(m-1)/2}|\overline{\{1^{d_j+m}};a_i+m\}|. \tag{4.9}$$

This is the analogue of (3.10), (3.17) and (3.18).

An exactly parallel procedure starting from the relation

$$\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=\epsilon^m\{\bar{\boldsymbol{\rho}}\} \tag{4.10}$$

with

$$\rho_i=\begin{cases}\nu_i+m & \text{for } i=1,2,\ldots,\tilde{\mu}_1\\ m & \text{for } i=\tilde{\nu}_1+1,\tilde{\nu}_1+2,\ldots,N-\tilde{\mu}_1\\ m-\mu_{N-i+1} & \text{for } i=N-\tilde{\mu}_1+1,N-\tilde{\mu}_1+2,\ldots,N\end{cases} \tag{4.11}$$

leads to the determinantal expansion

$$\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=(-1)^{m(m-1)/2}|\{\overline{c_i+m};1^{b_j+m}\}| \tag{4.12}$$

or equivalently

$$\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=(-1)^{m(m-1)/2}|\{\overline{\nu_i-i+m};1^{\tilde{\mu}_j-j+m}\}|. \tag{4.13}$$

In these expressions the ijth term is a character specified by a composite Young diagram consisting of a single hook formed from a row of dotted boxes and a column of undotted boxes.

5. Spinor characters and difference characters

The characters discussed in the previous sections do not fully exhaust the list of characters of irreducible representations of the classical groups. There also exist characters of the irreducible spinor representations of the full orthogonal group, $O(N)$. These may be denoted by $[\boldsymbol{\Delta}; \boldsymbol{\lambda}]$ where $\boldsymbol{\lambda}$ is a partition associated with a set of tensor indices and $\boldsymbol{\Delta}$ is associated with a spin index and denotes the basic spin character of $O(N)$ of dimension 2^k for $N = 2k+1$ and $N = 2k$.

The character formulae of Weyl (1926 § 5 (36), (40)) are such that

$$[\boldsymbol{\Delta}; \boldsymbol{\lambda}] = \boldsymbol{\Delta}|\{\lambda_i - i + j\} - \{\lambda_i - i - j + 1\}|. \tag{5.1}$$

It is possible to analyse this character exactly as above, making use of (3.5) to obtain the formula

$$[\boldsymbol{\Delta}; \boldsymbol{\lambda}] = \boldsymbol{\Delta}\left|\sum_{k=1}^{n} (-1)^{k+1}(\{\lambda_i - i + k\} - \{\lambda_i - i - k + 1\})\{1^{\tilde{\lambda}_j - j - k + 1}\}\right| \tag{5.2}$$

where as usual $i, j = 1, 2, \ldots, r$. Now, however, the ijth element of this determinant is not, in general, an irreducible character of $O(N)$.

It is possible to derive a simpler expression for those spinor characters of $O(N)$ corresponding to group elements having 1 as an eigenvalue. In such a case the symmetric functions $\{m\}$ of the complete set of eigenvalues may be rewritten as a sum of symmetric functions of the eigenvalues other than 1. The required identity is

$$\{m\}_N = \sum_{n=0}^{m} \{n\}_{N-1} \tag{5.3}$$

where subscripts have been added to emphasise the number of eigenvalues. Making use of this in (5.1) gives

$$[\boldsymbol{\Delta}; \boldsymbol{\lambda}]_N = \boldsymbol{\Delta}_N\left|\sum_{k=0}^{2j-2} \{\lambda_i - i + j - k\}_{N-1}\right|. \tag{5.4}$$

Subtracting successive columns then gives

$$[\boldsymbol{\Delta}; \boldsymbol{\lambda}]_N = \boldsymbol{\Delta}_N|\{\lambda_i - i + j\}_{N-1} + (1 - \delta_{1j})\{\lambda_i - i - j + 2\}_{N-1}|. \tag{5.5}$$

Comparison with (3.13) gives

$$[\boldsymbol{\Delta}; \boldsymbol{\lambda}]_N = \boldsymbol{\Delta}_N\langle\boldsymbol{\lambda}\rangle_{N-1}. \tag{5.6}$$

This result obtained by Bauer (1954) is valid for all the spinor characters of $O(2k+1)$, since for this group all elements possess the eigenvalue 1. The symplectic characters are then those of $Sp(2k)$. Even in the case of $O(2k)$ the result (5.6), as pointed out by Bauer, has some significance. Although no group $Sp(2k-1)$ exists, the characters $\langle\boldsymbol{\lambda}\rangle_{2k-1}$ may be defined through (3.13). Moreover the derivation of (3.20) proceeds as before even if $N = 2k-1$. Therefore

$$[\boldsymbol{\Delta}; \boldsymbol{\lambda}]_N = \boldsymbol{\Delta}_N|\langle\lambda_i - i + 1, 1^{\tilde{\lambda}_j - j}\rangle_{N-1}|. \tag{5.7}$$

In the case of the rotation group $SO(N)$ a complication is provided by the fact that the characters $[\boldsymbol{\lambda}]$ and $[\boldsymbol{\Delta}; \boldsymbol{\lambda}]$ of $SO(2k)$ are associated with reducible representations if $\boldsymbol{\lambda}$ is in the first case a partition into exactly k non-vanishing parts and not more than k

such parts in the second case. In these cases the reduction is such that

$$[\boldsymbol{\lambda}]=[\boldsymbol{\lambda}]_{+}+[\boldsymbol{\lambda}]_{-} \tag{5.8}$$

and

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]=[\boldsymbol{\Delta};\boldsymbol{\lambda}]_{+}+[\boldsymbol{\Delta};\boldsymbol{\lambda}]_{-}. \tag{5.9}$$

Correspondingly it is possible to define difference characters of SO($2k$) (Murnaghan 1938 pp 288, 315; Littlewood 1940 pp 246, 259)

$$[\boldsymbol{\lambda}]'=[\boldsymbol{\lambda}]_{+}-[\boldsymbol{\lambda}]_{-} \tag{5.10}$$

and

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]'=[\boldsymbol{\Delta};\boldsymbol{\lambda}]_{+}-[\boldsymbol{\Delta};\boldsymbol{\lambda}]_{-}. \tag{5.11}$$

These characters have been evaluated by Weyl (1926 § 5 (39), (40)) in the form

$$[\boldsymbol{\lambda}]'=[1^k]'|\{\lambda_i-i+j-1\}+(1-\delta_{1j})\{\lambda_i-i-j-1+2\}| \tag{5.12}$$

and

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]'=\boldsymbol{\Delta}'|\{\lambda_i-i+j\}+\{\lambda_i-i-j+1\}|. \tag{5.13}$$

Writing $\mu_i=\lambda_i-1$ for $i=1,2,\ldots,k$ and $[\boldsymbol{\lambda}]'=[\Box;\boldsymbol{\mu}]'$ with $\Box'=[1^k]'$, the first of these yields on comparison with (3.13)

$$[\Box;\boldsymbol{\mu}]'=\Box'\langle\boldsymbol{\mu}\rangle. \tag{5.14}$$

The second may be simplified by writing the symmetric functions $\{m\}$ in terms of the symmetric functions of the same set of eigenvalues together with an additional eigenvalue 1. The appropriate identity is the inverse of (5.3), namely

$$\{m\}_N=\{m\}_{N+1}-\{m-1\}_{N+1}. \tag{5.15}$$

Hence (5.13) yields

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]'_N=\boldsymbol{\Delta}'_N|\{\lambda_i-i+j\}_{N+1}-\{\lambda_i-i+j-1\}_{N+1}+\{\lambda_i-i-j+1\}_{N+1}-\{\lambda_i-i-j\}_{N+1}| \tag{5.16}$$

and adding successive columns gives

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]'_N=\boldsymbol{\Delta}'_N|\{\lambda_i-i+j\}_{N+1}-\{\lambda_i-i-j\}_{N+1}|, \tag{5.17}$$

so that from (3.12)

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]'_N=\boldsymbol{\Delta}'_N[\boldsymbol{\lambda}]_{N+1}. \tag{5.18}$$

This is the analogue of equation (5.6).

Reduced determinantal expansions follow from the application of equations (3.19) and (3.20) to (5.18) and (5.14), from which if required it is possible to write down those appropriate to $[\boldsymbol{\lambda}]_{\pm}$ and $[\boldsymbol{\Delta};\boldsymbol{\lambda}]_{\pm}$, thus completing the analysis of all the characters of irreducible representations of the classical groups.

The formulae obtained here are used in the following paper to derive formulae for the dimensions of each of the irreducible representations of the classical groups.

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